The Mathematics of Socionics

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I wrote an article for Wikisocion a few years ago about the math of socionics – which was destroyed in a database crash – and I'd like to reproduce the content here, with lots of new stuff too. It's a little abstract but I'll try to make it more accessible to newbies as I rewrite it. To my knowledge, most of this content has never been researched before, with the obvious exception of Reinin dichotomies. There are rumors of a Russian article, however.

The main field of relevance is finite group theory and group actions. If I am type t and F is a relationship, there is a type $F(t)$ that has that relationship with me. E.g. if F is beneficiary and t is SLI, $F(t)$ is ESI, SLI's beneficiary. (We have to distinguish between beneficiary and benefactor; a type's beneficiary and benefactor are not the same.) Therefore, a relationship is a function from the set of types to itself. There are 16 different relationships, which can be composed to make other relationships. E.g. dual * activator = mirror.

The basic rules for a group action are that $(F * G)(t) = F(G(t))$ and that $e(t) = t$ where e is the identity (identical) relationship.[1]

From now I will use the following abbreviations:

e	identity
d	dual
a	activator
m	mirror
g	superego
c	conflictor
q	quasi-identical
x	extinguishment
S	supervisor
B	benefactor
k	comparative
h	semidual (for 'half')
s	supervisee
b	beneficiary
ℓ	lookalike
i	illusionary

I picked some of the names strangely so they wouldn't clash.

[1] It's a math convention to call the identity e for eigen, which means "self" in German.

Structure of the group of relationships

The group of relationships in socionics, let's call it R, has a particular structure. For some relationships, like mirror, if you apply them twice you get the type you started with. But for others, like supervisee, you have to apply them four times to get back to the original type. So we say that s "has order 4". Every relationship in socionics has order 1, order 2, or order 4.[2]

R can be described fairly easily as a product of two other groups, D_4 and $\mathbb{Z}_2$. The latter is the group with two elements, e.g. addition mod 2 on the set $\{0, 1\}$, where $1 + 1 = 0$, $1 + 0 = 1$, and $0 + 0 = 0$. The former is the group of symmetries of a square, reflections and rotations. The rotation is what accounts for the asymmetric relations in socionics. The product means that we can think of relationships as pairs, the first element of the pair taken from D_4 and the second from $\mathbb{Z}_2$. So essentially we can visualize a type as two squares, which are simultaneously rotated or reflected, and can be switched with each other. This is just a way of describing Model A, with one square being the mental loop and the other the vital loop.

The Fundamental Data

R having the structure of $D_4 \times \mathbb{Z}_2$ is the first part of what we can call the Fundamental Data of Socionics. The second part of the data describe how the relationships relate to the types.

We want first of all, for there to be a relationship between any two types. And second, we want there to be **at most** one relationship between any two types. In terms of group actions the first condition says that the action is *transitive* and the second says that it's *free.* The two together mean that there is a **unique** relationship between any two types. And remember, we are thinking of relationships as functions, so that means for types s and t, there is exactly one relationship r such that $r(s) = t$.

These two requirements guarantee that there are the same number of types as relationships: 16.

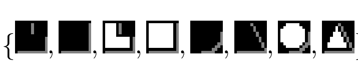
You may say, we're forgetting some additional structure on the set of types: the dichotomies. However, I will show that the standard system of dichotomies (as well as another, nonstandard one!) can actually be derived from the relationships. The group of relationships is the foundation of socionics, dichotomies are purely secondary.

This concludes the Fundamental Data of Socionics. The additional data are for Model A.

Model A

Model A includes the Fundamental Data, but adds on some more:

First, there is a set F of functions and a set

I = {Fe, Te, Fi, Ti, Se, Ne, Si, Ni = {  } = {△, ▲, □, ■, ◯, ●, ○, ●}

[2]Exercise: Which relationship has order 1?

of information elements. Traditionally the functions have an order and are identified with the set 12345678.[3] The order doesn't really mean anything, but I'll use the numbers as a convention. Then, R is constructed as a group of permutations of the functions, namely the one generated by $s = (1234)(5678)$, $d = (15)(26)(37)(48)$, and $a = (16)(25)(38)(47)$[4]. That is, $s(1) = 2$, $s(2) = 3$, etc. Now, R also acts via composition on the set of bijections from F to I[5] and we define the set of types T to be all rearrangements of, e.g., ILE = [Ne Ti Se Fi Si Fe Ni Te], which result from applying every permutation in R. Bijections always act freely by composition (cancellation law), and because of the way we defined T, R will act transitively, so we regain the Fundamental Data.

Model A is special, as we will see, but the way we've defined it there are many other possible models for socionics, i.e. representations of relationships as ways of permuting n functions (making R a subgroup of the group of permutations, S_n), and types as sequences of those n functions. Notice, when we are looking at models, R acts on both types *and* functions. It's important not to confuse these two actions.

Subgroups of R

It's helpful to know all the subgroups of R. A subgroup is a subset of the group that is also a group - i.e. contains the identity, inverses for all elements in it, and contains the product of any two elements. A particularly special subgroup of any group is the center Z (elements commuting with all other elements). In this case $Z(R) \cong Z(D_8) \times Z(\mathbb{Z}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, so $Z = \{e, g\}\{e, x\} = egxd$. Cosets of a subgroup are the sets you get when you multiply everything in the subgroup by some group element. Cosets of the center are

$egxd$
$bBsS$
$aqcm$
$ihkl$

Now, R can be completely described by taking the elements a, b, x (a, b for D_4 and x for $\mathbb{Z}_2$) and requiring the relations

$a^2 = b^4 = x^2 = e$,

$ax = xa, bx = xb$ (x commutes with both a and b), and

$(ab)^2 = e$, or $ab^{-1} = ba$.

It turns out a can be replaced with any relation from the last two cosets of Z, b can be replaced with any in the second coset, and x can be replaced only with d. The resulting set will still satisfy the same relations, and thus gives an automorphism of the group. These automorphisms will help us find the subgroups more easily.

So from now on I'll call the relations in the last two cosets **odd relations**.[6]

[3] Normally sets are written with commas and braces, but I will remove them when no ambiguity results.
[4] This is standard notation for permutations; see Wikipedia.
[5] Technically this is what's called a "right action", since we do tr instead of rt.
[6] To show that $(ab)^2 = e$ for any b of order 4 and a odd, it's enough to show that $|ab| \neq 4$. Well, notice that

List of subgroups

Lagrange's theorem says that a subgroup's size must divide the size of the whole group. So in our case they must have $1, 2, 4, 8$, or 16 elements.

First, there are the subgroups $\{e\}$ and the whole group, as in any group. The subgroups of order 2 are those generated by one of the 11 relationships of order 2. The subgroups of order 4 are either generated by a relationship of order 4:

$$ebgB, esgS$$

or only have elements of order 2:[7]

Elements	Description	Normal?
$egxd$	$Z =$ Dem/Arist ∩ J/P	yes
$egaq$	I/E ∩ Dem/Arist	yes
$egcm$	Stat/Dyn ∩ Dem/Arist	yes
$egih$	J/P ∩ Neg/Pos	yes
$egk\ell$	temperament=I/E ∩ J/P	yes
$exac$	Dem/Arist ∩ Emot/Const	no
$exqm$	club	no
$exik$	J/P ∩ T/F	no
$exh\ell$	J/P ∩ N/S	no
$edam$	quadra	no
$edqc$	Dem/Arist ∩ Far/Care	no
$edi\ell$	2nd value same	no
$edhk$	1st value same	no

All these intersect the center non-trivially because if r, s have order 2 and aren't in the center, then they are odd, so $rsb = brs$, and $rss = srs$ because odd elements conjugate each other trivially. [WHY?] [8]

Subgroups of order 8 correspond to dichotomies. It turns out if you take a particular type and apply all of the 8 relations in the subgroup, you'll get 8 types that share a particular Reinin

the set of odd elements is the complement of the group generated by the elements of order 4, so if $|ab| = 4$, $a = abb^{-1}$, a contradiction.

[7] Hence, isomorphic to $\mathbb{Z}_2^2$.

[8] Technical note: It turns out that a subgroup is normal iff it is contained in the center or it contains g. The $\Longrightarrow$ implication follows as R is a p-group. As for the rest, if x is conjugate to y and $x \neq y$, $x = gy$, so $xy^{-1} = g$.

dichotomy.

Elements	Description	Type
$egxdbBsS$	Result/Process	$\mathbb{Z}_4 \times \mathbb{Z}_2$
$egxdaqcm$	Democratic/Aristocratic	$\mathbb{Z}_2^3$
$egxdihk\ell$	Rational/Irrational	$\mathbb{Z}_2^3$
$egaqbBk\ell$	Introverted/Extroverted	D_4
$egaqsSih$	Negativist/Positivist	D_4
$egcmbBih$	Questioner/Declarer	D_4
$egcmsSk\ell$	Static/Dynamic	D_4

So for example, if two types are both Process or both Result, they will necessarily have 1 of the 8 relationships $egxdbBsS$; and if one is Process and the other Result they'll have one of the other 8 relationships. [9]

Notice that the dichotomies represented here are only the ones that Superego partners share, because they all contain the Superego relation. As for why *all* such 7 dichotomies are represented as subgroups – well, see next section.

Type dichotomies

A dichotomy system is a way of describing each type as a set of binary choices, e.g. a vector like (I, S, T, P). However, the set of types is not really a vector space because a vector space has a way of adding vectors and a special element 0 such that $v + 0 = v$ for all v. (In 'introverted socionics' the set of types has a 0.) We don't want to be able to "add" types together to get other types. The way to avoid creating an addition operation on the types is to instead think about how one can "subtract" types to get a vector of 0s and 1s, where 1 means a flip and 0 means no flip. This means that the set of types is actually acted on by the vector space $\mathbb{Z}_2^4$.[10] This fits in nicely with our description of intertype relationships, and actually about half of the vectors line up with relationships. E.g., the vector (1, 0, 0, 0) represents extinguishment.[11]

Again we can get the equal divisions of the set of types by applying a sub-vector space of size 8 of the vector space of dichotomies.

Now I will show how we can derive the dichotomies from the relationships. One can obtain *most* of the system by taking the subgroup D Democratic/Aristocratic (i.e. the relationships that preserve this dichotomy), since these relationships all act by flipping certain Jungian dichotomies. The rest, however, do not, so we only have 8 of the 16 total vectors needed. Therefore we need to add another permutation z that isn't strictly based on relationships. Once we have that, we can multiply it by (add it to) the 8 others to get 16 in all. We can add, for example,

$$z(t) = \begin{cases} B(t) & \text{if t is rational} \\ b(t) & \text{else} \end{cases}$$

[9] Technical note: All of these are normal because they have index 2. The last four are isomorphic to D_4 because they have an element of order 4 and aren't abelian.

[10] This action is manifestly free and transitive.

[11] This approach may seem to depend on the Jungian basis, because of the way we interpret the vectors, but is actually independent of it. More precisely one should say that the Reinin dichotomies left the same by (1, 0, 0, 0) are those generated by N/S, T/F, and J/P.

This corresponds to flipping T/F and J/P in the Jungian basis, and in combination with everything in D will generate all the rest of the dichotomy flips.

To verify that this makes a 4-dimensional vector space we prove (1) that z has order 2 and (2) that z commutes with everything in D. For (1), if t is rational, $B(t)$ is irrational so $z(z(t)) = z(B(t)) = b(B(t)) = t$, and similarly if t is irrational. Thus $z^2 = e$.

Now for commutativity. r flips rationality iff r is odd. If r is odd, $rB = br$, and if r isn't odd, r commutes with B.

	r odd	r even
t rational	$rz(t) = rB(t) = br(t) = zr(t)$	$rz(t) = rB(t) = Br(t) = zr(t)$
t irrational	$rz(t) = rb(t) = Br(t) = zr(t)$	$rz(t) = rb(t) = br(t) = zr(t)$

There is also a way to redefine the group operation and group action so that B itself flips the T/F and J/P dichotomies. The resulting function will agree with benefactor on the rational types and with beneficiary on the irrational types. This is a little abstract; see appendix.

The underlying reason that we can use the subgroup D is that it has the same structure as $\mathbb{Z}_2^3$–a $\mathbb{Z}_2$ vector space, which is simply a group in which every element has order 2. This same fact is only true of one other subgroup of order 8: Rational/Irrational (J for short). And in fact, we can use an exactly parallel function to complete that subgroup to a dichotomy system:

$$z'(t) = \begin{cases} B(t) & \text{if t is democratic} \\ b(t) & \text{else} \end{cases}$$

Notice how all we did is switch J and D. We can use an identical proof to verify that this creates a complete dichotomy system. Now, what do these new dichotomies represent? We need to work out some of the orbits of dimension-3 subspaces. For example, d, k, z' gives

LII ESE LSI EIE IEI SLE ILI SEE

LIE ESI LSE EII IEE SLI ILE SEI

as our two sets.

Another example: d, ℓ, z' gives

LII ESE EII LSE IEI SLE SEI ILE

LIE ESI EIE LSI IEE SLI SEE ILI

Do you see the pattern?

The first dichotomy is base function Beta vs. base function Delta, and the second is creative function Alpha vs. Gamma. We can complete these to a basis with I/E (k, ℓ, z') and J/P. Notice that those two are also based on dichotomous properties of the type's functions, and hence, so are all of the rest. There are 8 information elements, so there are 8 - 1 = 7 dichotomies of

information. Whether the first function is I/E, J/P, static/dynamic determines the same for the second. So we get:

4: 1st function alpha/gamma, beta/delta, external/internal, involved/abstract

4: 2nd function alpha/gamma, beta/delta, external/internal, involved/abstract

4: I/E, J/P, static/dynamic, democratic/aristocratic

3: questioner/declarer, negativist/positivist, process/result

= 15 in all, just like the Reinin dichotomies.

Other dichotomies

Now, in Model A F and I have size 2^3 so we can construct dichotomy systems on them, call them F^* and I^*. R already acts on F, so we will simply define $F^* = D$. D is a vector space and it acts transitively on F (think, e.g., Ni can occur in any position in an Aristocratic type). If a certain function occurs in the same place for two Democratic (or Aristocratic) types, they must be the same. Hence, the action is faithful.[12] Dichotomies are

Subgroup	Dichotomy	1st set	2nd set
quadra	valued/subdued	1256	3478
$egxd$	accepting/producing	1357	2468
$egaq$	"version"	1368	2457
$egcm$	conscious/unconscious	1234	5678
$exac$		1467	2358
$edqc$	evaluatory/situational	1458	2367
club	strong/weak	1278	3456

There is again an alternative given by J. [...]

I^* is more tricky. Z acts on I in the obvious way.[13] That gives us "two of the three dichotomies", so to speak. And, remarkably enough, the last is given by z' above! We obtain

Subgroup	Dichotomy	1st set	2nd set
$egxd$	irrational/rational	$NeSeNiSi$	$TiFiTeFe$
e, g, z', gz'	extroverted/introverted	$NeSeTeFe$	$TiFiNiSi$
e, x, z', xz'	abstract/involved	$NeNiTeTi$	$FiFeSiSe$
e, d, z', dz'	Delta/Beta	$NeTeSiFi$	$NiTiSeFe$
e, d, xz', gz'	Alpha/Gamma	$NeSiFeTi$	$TeFiSeNi$
e, x, gz', dz'	internal/external	$NeNiFeFi$	$TiTeSiSe$
e, g, xz', dz'	static/dynamic	$NeSeTiFi$	$TeFeNiSi$

[12]Technical note: or, you can just check $|D| = |F|$.
[13]Explicitly, $r(Xa) = r(t)(t^{-1}(Xa))$ for any t

Models

The smallest models of R will not be faithful (i.e. there will be less than 16 types), but let's classify them anyways. We will only consider models that have no fixed points, since if they do, they can be considered as acting on a smaller set.

A group action splits up the set into partitions called *orbits*. Functions x and y are said to be in the same orbit if there is a relationship r such that $r(x) = y$. This is an equivalence relation; if $r(x) = y$ and $s(y) = z$, then $sr(x) = z$. It turns out that we can consider each orbit of a model separately. And whenever we have two models, we can set them side-by-side to create a composite model. Therefore we need only classify the models with one orbit–i.e., the *transitive actions*.

So, a model is transitive when for all functions i and j in the model, there is a relationship r such that $r(i) = j$. Model A is a transitive model of order 8. Transitivity puts all the functions on the same footing, so to speak. The models of order 6, for example, are not transitive: they have two unbreachable classes of functions (e.g., NTSF and IE).

We will also ignore models with fixed points (orbits of size 1), since we can remove them.

Orbits of R must have size dividing the size of R, 16. So if the action is transitive, obviously n is even and ≤ 16.

Order 2

If R acts transitively on a set of size 2, the kernel of the homomorphism from R to S_2 satisfies $|\ker(\alpha)| = 8$. Hence there is a model for each subgroup of order 8 (since all are normal). Here the "types" are very coarse: they are just single dichotomies, and the "functions" are the two dichotomous traits, arranged in order of preference.

Order 4

For S_4 (size 24), $|\ker(\alpha)| \geq 2$, because $\gcd(16, 24) = 8$.

If $|\ker(\alpha)| = 2$, the only possibilities for the kernel are eg, ex, and ed, whose quotients are $\mathbb{Z}_2^3$[14], D_4[15], and D_4[16]. It is clear that $\mathbb{Z}_2^3$ doesn't embed into S_4 (using the below lemmas), but D_4 does (thinking of the four points as corners of a square). The rotation in D_4 can be made to go to (1234), and the possibilities for the flip are given by the proof below for S_6; all are in the standard representation of D_4 where $a = (12)(34)$, for example. Thus, there are two unique transitive models, viz.:

$NTSF$ for ILE and ILI

and

[14] proof: no element of order 4

[15] The only group of order 8 which is not abelian; the quotient is not abelian because $[ab]$ and $[ba]$ are different.

[16] Same as the previous by automorphism of R.

$\alpha\beta\gamma\delta$ for ILE and SEI

where it is understood that if the quadras go in reverse order, the type is a Result type.[17] Let's call these models *loops*.

So, what this tells us is that Extinguishers and Duals both have something very much in common in their relationships. We can surmise that the former has to do with strengths and the latter with quadra values.

For $|\ker(\alpha)| = 4$, there is at least one model for every normal subgroup of order 4, namely the 5 containing g.[18] For all, the group of coarse relationships (the quotient and image in S_4) has no element of order 4, as $[b^2] = [g] = [e]$ and same with s. Thus it is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, and is generated by two commuting elements of order 2, which one can pick to be of the form $(12), (34)$ (see below lemmas for why). Thus the type model is just a pair of dichotomies, as listed above. Thus, it is the union of two models on S_2.

It may be worthwhile describing interactions between these coarse types, an example of which is temperaments.

Order 6

For order 6, there can be no transitive model, since 6 is not a power of 2. Therefore any model of this size is a collection of smaller models. It can be either:

• 3 dichotomies. Notice the only dichotomies allowed are ones that superego partners share; therefore, the only non-redundant choices make Superegos the same type.

• A dichotomy and a loop. If you pick a dichotomy to distinguish duals or extinguishers (depending on the loop), you get a faithful model with 16 types, such as $\alpha\beta\gamma EI$, $NTSF+-$, etc.

It is easier for this proof to number the "functions" 012345.

Elements in R have order 2^i for some i, so they are always represented as products of cycles with length 2^j where $\gcd(j) = \max(j) = i$.

Lemma. 1 If $gh = hg$ and $h^n(x) = x$, $g(x) = gh^n(x) = h^n g(x)$. That is, commuting elements preserve "order" of elements in the set acted on; they can't move elements between orbits of different sizes, and in particular must preserve fixpoints (orbits of size 1).

It turns out that the same holds if $gh = h^{-1}g$, since $g(x) = gh^n(x) = h^{-n}g(x)$ so $h^n g(x) = g(x)$. Let's say that in either of these situations "h forces g". Thus any element of order 4 forces every other element, and so does any element in the center.

[17] This must be where the name Left/Right comes from. If b and s both correspond to a simple shift, then the "orientation" of the functions must correspond to Process/Result, the group generated by b and s. We can also construct these models from Model A by identifying "opposite-version' functions or dual functions, respectively, in which case the interpretation of the latter model changes somewhat, e.g.: Reasonable Merry Resolute Serious. But notice that those values apply to α, β, γ, and δ, in that order.

[18] The subgroup must be normal, which means that it is impossible to construct relationships between clubs, for example, because no club can consistently be called the "supervisor" of another club. One NT's supervisor is an ST, while another's is an NF.

And for any r in R, either $rb = br$ or $rb = b^{-1}r$ (proof by checking cosets of Z). Therefore, for example, a must preserve the elements of the 4-cycle of s above.

Lemma. 2 r commutes with $(0123) = [x+1 \mod 4] \implies r(d+1) = r(d)+1$. Thus, by induction, $r(d) = d+k$ for some k. $k = 1, 3$ correspond to (0123) and its inverse respectively. So if r has order 2 it must be +2 or +0 (identity) on that segment.

Now we just have to show that a is determined up to isomorphism.

Remember $ar = r^{-1}a$. So on 0123, $a(n+1) = a(n)-1$. By induction, this just means $a(n) = k-n$ for some k.

$k = 0$: (13)

$k = 1$: (01)(23)

$k = 2$: (02)

$k = 3$: (03)(12)

Order 8

Here again we can place smaller models side by side to get the non-transitive models. The possibilities are:

- Four dichotomies,
- One "loop" and two dichotomies (redundant addition to the models of order 6),
- Two loops: $NTSF\alpha\beta\gamma\delta$. This is another faithful model which is not transitive.

So far we haven't seen any models that are both faithful *and* transitive. But of course, Model A is such a model.

If the model is faithful and transitive, $n = 8$ or 16. If $n = 16$, the action must be free as well, by the pigeonhole principle: if $Rx = S$, then application to x is surjective, so it is injective because R is finite. Thus any transitive model of order 16 is isomorphic to R's action on T.[19] I will show that the only other transitive, faithful model is Model A.

Proof. If any element of order 4 in R has orbits of two different sizes, then it forces all the other elements to also not join those two orbits. Therefore, if the action is transitive, the elements of order 4 are all products of 2 4-cycles. So WLOG, $S = (1234)(5678)$. Now, consider what elements d of order 2 commute with S: if $d(m) = n$ where m and n are in two different cycles of S, $d(m+1) = dS(m) = Sd(m) = S(n) = n+1$, so by induction d must be of the form (15)(26)(37)(48) (up to a permutation of F).[20] If d does not cross the orbits of S, the lemmas above tell us that d must be (13)(24), (57)(68), or (13)(24)(57)(68). The latter is $S^2 = g$– impossible. And d can't be either of the two former, because if it was, it would force everything to not cross (1234) and (5678).

Finally, we determine a, an arbitrary odd element. If a crosses the orbits of S, $a(m+1) = aS(m) = sa(m) = s(n) = n-1$, so by induction a is of the form (15)(28)(37)(46) (4 possibilities). And if it doesn't,

[19] I believe Gulenko originated a 16-function model with Fi+, Fi-, etc. But one might as well say an ILE's leading function is not Ne+, but ILE!

[20] d in Model A; x in Model A is thus generated as B^2d.

a must commute with x. It is not possible for a to not cross an orbit of x (because then $a = x$), so $a(m) = n$ for m, n in different orbits, it must have $a(x(m)) = x(n)$. So a is either (12)(56)(34)(78), (13)(57)(24)(68), ... There are 3*2*2=12 possibilities in all.

to be continued